NOTES ON THE OCTAHEDRAL AXIOM

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Abstract. In this self-contained note, we consider a pre-triangulated category and consider several possible additional axioms. We then prove that they are all equivalent to the Octahedral Axiom. The main reference is of course [1].

1. Pre-Triangulated Categories

Let $\mathcal{T}$ be an additive category with an autoequivalence $[1]\colon \mathcal{T} \to \mathcal{T}$. A triangle in $\mathcal{T}$ is a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ and a morphism of triangles $[x,y,z]\colon (f,g,h) \to (f',g',h')$ is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle x[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

We call (f,g,h) a pre-triangle provided that each functor $\mathrm{Hom}(U,-)$ yields a long-exact sequence of abelian groups

$$\cdots \to (U,X) \xrightarrow{f} (U,Y) \xrightarrow{g} (U,Z) \xrightarrow{h} (U,X[1]) \xrightarrow{f[1]} (U,Y[1]) \to \cdots$$

where we have used the short-hand $(U,X) := \mathrm{Hom}(U,X)$.

The following consequences are immediate:

(1) Direct sums and direct summands of pre-triangles are again pre-triangles.
(2) Each $0 \to X = X \to 0$ is a pre-triangle.
(3) (f,g,h) is a pre-triangle if and only if $(-g,-h,-f[1])$ is a pre-triangle.
(4) If (f,g,h) is a pre-triangle, then $gf = 0$.

The next lemma will be used frequently.

Lemma 1. *If $[x,y,z]$ is a morphism of pre-triangles such that two of x, y and z are isomorphisms, then so is the third.*

Proof. Suppose that x and y are isomorphisms. The for each U we have an exact commutative diagram of abelian groups

$$\begin{array}{ccccccccc} (U,X) & \xrightarrow{f} & (U,Y) & \xrightarrow{g} & (U,Z) & \xrightarrow{h} & (U,X[1]) & \xrightarrow{f[1]} & (U,Y[1]) \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle x[1]} & & \downarrow{\scriptstyle y[1]} \\ (U,X') & \xrightarrow{f'} & (U,Y') & \xrightarrow{g'} & (U,Z') & \xrightarrow{h'} & (U,X'[1]) & \xrightarrow{f'[1]} & (U,Y'[1]) \end{array}$$

By the Five Lemma, $z\colon \mathrm{Hom}(U,Z) \to \mathrm{Hom}(U,Z')$ is an isomorphism for each U. Setting $U = Z'$, there exists z' such that $zz' = \mathrm{id}_{Z'}$. Setting $U = Z$, we have $z(\mathrm{id}_Z - z'z) = 0$, hence $z'z = \mathrm{id}_Z$. $\square$

1.1. **Definition.** We fix a collection of triangles in $\mathcal{T}$, closed under isomorphism, which we call distinguished or exact. We say that $\mathcal{T}$ is a pre-triangulated category with respect to the exact triangles provided that

TR0 Each $0 \to X \xrightarrow{1} X \to 0$ is exact.
TR1 Each $f\colon X \to Y$ can be extended to an exact triangle (f, g, h).
TR2 (f, g, h) is exact if and only $(-g, -h, -f[1])$ is exact.
TR3 If (f, g, h) and (f', g', h') are exact triangles, then any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & & & \downarrow{\scriptstyle x[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

can be completed to a morphism of triangles.

1.2. **Consequences.**

Lemma 2. *Each exact triangle is a pre-triangle. In other words, each functor* $\mathrm{Hom}(U, -)$ *is homological. Dually each functor* $\mathrm{Hom}(-, U)$ *is cohomological.*

Proof. It is enough to show that we have an exact sequence

$$(U, X) \xrightarrow{f} (U, Y) \xrightarrow{g} (U, Z)$$

To see that $gf = 0$, apply TR3 to the exact commutative diagram

$$\begin{array}{ccccccc} X & = & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \| & & \downarrow{\scriptstyle f} & & & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

Suppose $\theta\colon U \to Y$ satisfies $g\theta = 0$. Applying TR3 to

$$\begin{array}{ccccccc} U & = & U & \longrightarrow & 0 & \longrightarrow & U[1] \\ & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

shows that θ factors through f. □

Lemma 3. *Direct sums of exact triangles are exact.*

Proof. Let (f, g, h) and (f', g', h') be exact triangles. By TR1, complete the map $f \oplus f'$ to an exact triangle $(f \oplus f', c, d)$. By TR3, we can complete the commutative diagram

$$\begin{array}{ccccccc} X \oplus X' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & Y \oplus Y' & \xrightarrow{c} & C & \xrightarrow{d} & X[1] \oplus X[1] \\ \downarrow{\scriptstyle (1\ 0)} & & \downarrow{\scriptstyle (1\ 0)} & & & & \downarrow{\scriptstyle (1\ 0)} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \end{array}$$

to a morphism of triangles via some $z\colon C \to Z$. Similarly for (f', g', h'). We thus have a commutative diagram

$$\begin{array}{ccccccc}
X \oplus X' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & Y \oplus Y' & \xrightarrow{c} & C & \xrightarrow{d} & X[1] \oplus X'[1] \\
\| & & \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)} & & \| \\
X \oplus X' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & Y \oplus Y' & \xrightarrow{\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)} & Z \oplus Z' & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)} & X[1] \oplus X'[1]
\end{array}$$

Each row is a pre-triangle, so we can apply Lemma 1 to deduce that $\left(\begin{smallmatrix} z & 0 \\ 0 & z' \end{smallmatrix}\right)$ is an isomorphism, and hence that the bottom row is exact. □

Lemma 4. *Direct summands of exact triangles are exact.*

Proof. Suppose that the direct sum $(f, g, h) \oplus (f', g', h')$ is exact. Then (f, g, h) is a pre-triangle, and we need to show that it is exact. By TR1 we have an exact triangle (f, c, d) and by TR3 we can complete the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{c} & C & \xrightarrow{d} & X[1] \\
\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} & & & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \\
X \oplus X' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} & Y \oplus Y' & \xrightarrow{\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)} & Z \oplus Z' & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)} & X[1] \oplus X[1]
\end{array}$$

to a morphism of triangles via some $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) : C \to Z \oplus Z'$. We now have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{c} & C & \xrightarrow{d} & X[1] \\
\| & & \| & & \downarrow{\scriptstyle \alpha} & & \| \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1]
\end{array}$$

Since the rows are pre-triangles, we can apply Lemma 1 to conclude that α is an isomorphism. Since the top row is exact, so is the bottom row. □

Lemma 5. *Suppose we have an exact triangle of the form*

$$X \xrightarrow{\left(\begin{smallmatrix} a \\ f \end{smallmatrix}\right)} A \oplus Y \xrightarrow{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} A \oplus Z \xrightarrow{(c\ h)} X[1].$$

If $\alpha \in \mathrm{Aut}(A)$*, then this is isomorphic to the direct sum of* $0 \to A = A \to 0$ *and* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$*, where* $g = \delta - \gamma\alpha^{-1}\beta$*. In particular,* (f, g, h) *is exact.*

Proof. Since the composition of maps in an exact triangle is zero, we have that

$$\alpha a + \beta f = 0 \quad \text{and} \quad c\alpha + h\gamma = 0.$$

This yields the isomorphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{\left(\begin{smallmatrix} a \\ f \end{smallmatrix}\right)} & A \oplus Y & \xrightarrow{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)} & A \oplus Z & \xrightarrow{(c\ h)} & X[1] \\
\| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{smallmatrix}\right)} & & \| \\
X & \xrightarrow{\left(\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right)} & A \oplus Y & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & g \end{smallmatrix}\right)} & A \oplus Z & \xrightarrow{(0\ h)} & X[1]
\end{array}$$

where $g := \delta - \gamma\alpha^{-1}\beta$. □

A commutative square

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

is called homotopy Cartesian if there is an exact triangle of the form

$$Y \xrightarrow{\binom{y}{-g}} Y' \oplus Z \xrightarrow{(g'\ z)} Z' \xrightarrow{\delta} Y[1].$$

We call δ the differential.

Important. We note that our exact triangle is isomorphic to

$$Y \xrightarrow{\binom{-y}{g}} Y' \oplus Z \xrightarrow{(g'\ z)} Z' \xrightarrow{-\delta} Y[1].$$

Thus the differential depends on where we choose to place the minus sign.

Similarly the differential also depends on how we draw the square. For, the square

$$\begin{array}{ccc} Y & \xrightarrow{y} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow{z} & Z' \end{array}$$

is homotopy Cartesian, but with differential $-\delta$.

To avoid confusion, we shall always describe the triangle arising from a homotopy Cartesian square.

2. Additional Axioms

We now introduce some further possible axioms, and which we show are all equivalent in the next section. A pre-triangulated category $\mathcal{T}$ is called triangulated if it satisfies any of these extra axioms. We briefly describe the axioms, before giving the precise formulation.

The first axiom, Axiom A, shows that the morphism in TR3 can be chosen such that the mapping cone is exact. In particular, we see that TR3 is now a consequence of Axiom A.

The second two, Axioms B and B', are special cases of Axiom A when one of the maps is known to be an isomorphism. Axiom B can be thought of as analogous to the existence of push-outs in an abelian category, and Axiom B' to the existence of pull-backs.

Axiom C is a kind of converse to Axiom B, and can be thought of as analogous to the fact that parallel maps in a pull-back/push-out square have isomorphic kernels and cokernels.

The fifth axiom, Axiom D, is often known as the Octahedral Axiom, since one can picture the diagram as an octahedron as follows. We first collapse all the identities to obtain a diagram with four triangles and one square. We then remove all copies of the shift [1], but adding a plus sign to an arrow if it was of the form $Z \to X[1]$. We now fold the triangles on the left and right of the square up, to meet in a point. Similarly, we fold the other two triangles underneath the square. This gives an octahedron such that

- the four exact triangles correspond to four non-adjacent faces of the octahedron;
- the other four faces are given by the commutative squares containing an identity map (repressed in the octahedron);
- the three slices of the octahedon forming squares give the commutative square of the original diagram, the relation $f[1]h' = wv'$ and an oriented cycle.

Axiom D' is a seemingly stronger statement providing an additional exact triangle.

Axiom E is a double application of Axiom E, removing the obvious bias in Axiom C given by choosing an orientation of the homotopy Cartesian square.

Axiom A. *Given the commutative diagram with exact rows*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & & & \downarrow{\scriptstyle x[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

there exists a morphism of triangles $[x, y, z]$ *such that the mapping cone is exact:*

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -g & 0 \\ y & f' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -h & 0 \\ z & g' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -f[1] & 0 \\ x[1] & h' \end{pmatrix}} Y[1] \oplus X'[1].$$

Axiom B. *Given the commutative diagram with exact rows*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \| & & \downarrow{\scriptstyle y} & & & & \| \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1] \end{array}$$

there exists a morphism of triangles $[1, y, z]$ *such that the following triangle is exact:*

$$Y \xrightarrow{\begin{pmatrix} y \\ -g \end{pmatrix}} Y' \oplus Z \xrightarrow{(g' \ z)} Z' \xrightarrow{f[1]h'} Y[1].$$

Axiom B'. *Given the commutative diagram with exact rows*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \| & & & & \downarrow{\scriptstyle z} & & \| \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1] \end{array}$$

there exists a morphism of triangles $[1, y, z]$ *such that the following triangle is exact:*

$$Y \xrightarrow{\begin{pmatrix} y \\ -g \end{pmatrix}} Y' \oplus Z \xrightarrow{(g' \ z)} Z' \xrightarrow{f[1]h'} Y[1].$$

Axiom C. *Given an exact triangle*

$$Y \xrightarrow{\begin{pmatrix} y \\ -g \end{pmatrix}} Y' \oplus Z \xrightarrow{(g' \ z)} Z' \xrightarrow{\delta} Y[1]$$

there exists a commutative diagram with exact rows

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \| & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \| \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1] \end{array}$$

such that $\delta = f[1]h$. *Moreover, we may fix beforehand either* (f, g, h) *or* (f', g', h').

Axiom D. *Given three exact triangles* (f, g, h), (f', g', h') *and* (u, v, w) *such that* $f' = uf$, *there exists a fourth exact triangle* (u', v', w') *fitting into a commutative diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \| \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1] \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} & & \\ & & W & = & W & & \\ & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \\ & & Y[1] & \xrightarrow{g[1]} & Z[1] & & \end{array}$$

and such that $f[1]h' = wv'$.

Axiom D'. *Under the same conditions as Axiom D, we may further assume that the triangle*

$$Y \xrightarrow{\binom{y}{-g}} Y' \oplus Z \xrightarrow{(g'\ z)} Z' \xrightarrow{\delta} Y[1]$$

is exact, where $\delta := f[1]h' = wv'$.

Axiom E. *Given an exact triangle*

$$Y \xrightarrow{\binom{u}{-g}} Y' \oplus Z \xrightarrow{(g'\ u')} Z' \xrightarrow{\delta} Y[1]$$

there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \| \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1] \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} & & \\ & & W & = & W & & \\ & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w'} & & \\ & & Y[1] & \xrightarrow{g[1]} & Z[1] & & \end{array}$$

such that $\delta = f[1]h' = wv'$. *Moreover, we may fix beforehand either* (f, g, h) *or* (f', g', h') *and either* (u, v, w) *or* (u', v', w').

3. Equivalence of the additional axioms

3.1. A implies B. Suppose that A holds and that we are given a diagram as in B. We may choose z such that the mapping cone is exact, and thus the following triangle is exact:

$$X\oplus Y \xrightarrow{\begin{pmatrix}0 & -g\\ f' & y\end{pmatrix}} Z\oplus Y' \xrightarrow{\begin{pmatrix}-h & 0\\ z & g'\end{pmatrix}} X[1]\oplus Z' \xrightarrow{\begin{pmatrix}1 & h'\\ -f[1] & 0\end{pmatrix}} X[1]\oplus Y[1].$$

Thus by Lemma 5 this has the exact triangle

$$Y \xrightarrow{\begin{pmatrix}-g\\ y\end{pmatrix}} Z\oplus Y' \xrightarrow{(\,z\ g'\,)} Z' \xrightarrow{f[1]h'} Y[1]$$

as a direct summand. Hence B holds.

3.2. B implies A. Suppose that B holds and that we are given a diagram as in A. We begin by considering the following isomorphism of triangles:

$$\begin{array}{ccccccc}
X\oplus X' & \xrightarrow{(\,0\ f'\,)} & Y' & \xrightarrow{\begin{pmatrix}0\\ g'\end{pmatrix}} & X[1]\oplus Z' & \xrightarrow{\begin{pmatrix}-1 & 0\\ 0 & h'\end{pmatrix}} & X[1]\oplus X'[1]\\
\downarrow{\scriptstyle\begin{pmatrix}1 & 0\\ -x & 1\end{pmatrix}} & & \| & & \| & & \downarrow{\scriptstyle\begin{pmatrix}1 & 0\\ -x[1] & 1\end{pmatrix}}\\
X\oplus X' & \xrightarrow{(\,yf\ f'\,)} & Y' & \xrightarrow{\begin{pmatrix}0\\ g'\end{pmatrix}} & X[1]\oplus Z' & \xrightarrow{\begin{pmatrix}-1 & 0\\ x[1] & h'\end{pmatrix}} & X[1]\oplus X'[1]
\end{array}$$

Since the top row is exact, so is the bottom row. We can thus form the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
X\oplus X' & \xrightarrow{\begin{pmatrix}f & 0\\ 0 & 1\end{pmatrix}} & Y\oplus X' & \xrightarrow{\begin{pmatrix}g\\ 0\end{pmatrix}} & Z & \xrightarrow{\begin{pmatrix}h\\ 0\end{pmatrix}} & X[1]\oplus X'[1]\\
\| & & \downarrow{\scriptstyle(\,y\ f'\,)} & & & & \|\\
X\oplus X' & \xrightarrow{(\,yf\ f'\,)} & Y' & \xrightarrow{\begin{pmatrix}0\\ g'\end{pmatrix}} & X[1]\oplus Z' & \xrightarrow{\begin{pmatrix}-1 & 0\\ x[1] & h'\end{pmatrix}} & X[1]\oplus X'[1]
\end{array}$$

Applying B, there exists $\begin{pmatrix}-a\\ z\end{pmatrix} : Z \to X[1]\oplus Z'$ completing this diagram to a morphism of triangles and such that the following triangle is exact:

$$Y\oplus X' \xrightarrow{\begin{pmatrix}-g & 0\\ y & f'\end{pmatrix}} Z\oplus Y' \xrightarrow{\begin{pmatrix}-a & 0\\ z & g'\end{pmatrix}} X[1]\oplus Z' \xrightarrow{\begin{pmatrix}-f[1] & 0\\ x[1] & h'\end{pmatrix}} Y[1]\oplus X'[1].$$

We note that $\begin{pmatrix}h\\ 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ x & h'\end{pmatrix}\begin{pmatrix}-a\\ z\end{pmatrix}$, and hence that $a = h$. Hence A holds.

3.3. B implies C. Suppose that B holds and that we are given a diagram as in C. Suppose further that we fix the triangle (f, g, h). We now form the commutative diagram with exact rows

$$\begin{array}{ccccccc}
Y & \xrightarrow{\begin{pmatrix}y\\ -g\end{pmatrix}} & Y'\oplus Z & \xrightarrow{(\,g'\ z\,)} & Z' & \xrightarrow{\ \delta\ } & Y[1]\\
\| & & \downarrow{\scriptstyle(\,0\ 1\,)} & & & & \|\\
Y & \xrightarrow{-g} & Z & \xrightarrow{-h} & X[1] & \xrightarrow{-f[1]} & Y[1]
\end{array}$$

We can thus find $-h' \colon Z' \to X[1]$ giving a morphism of triangles and such that the following triangle is exact:

$$Y' \oplus Z \xrightarrow{\begin{pmatrix} 0 & 1 \\ -g' & -z \end{pmatrix}} Z \oplus Z' \xrightarrow{(-h \ -h')} X[1] \xrightarrow{\begin{pmatrix} -(yf)[1] \\ (gf)[1] \end{pmatrix}} Y'[1] \oplus Z[1].$$

We note that $\delta = f[1]h'$ and $h = h'z$.

By Lemma 5, this has as direct summand the exact triangle

$$Y' \xrightarrow{-g'} Z' \xrightarrow{-h'} X[1] \xrightarrow{-(yf)[1]} Y'[1].$$

Rotating the triangle and setting $f' := yf$ yields the required commutative diagram with exact rows.

Suppose instead that we had fixed (f', g', h'). We apply the above construction to obtain triangles $(\bar{f}, g, \bar{h})$ and $(\bar{f}', g', \bar{h}')$ satisfying Axiom C. By TR3 and Lemma 1 there exists $x \in \mathrm{Aut}(X)$ such that $[x, 1, 1] \colon (\bar{f}', g', \bar{h}') \to (f', g', h')$ is an isomorphism of triangles. We define (f, g, h) such that $[x, 1, 1] \colon (\bar{f}, g, \bar{h}) \to (f, g, h)$ is also an isomorphism of triangles; that is, we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle x^{-1}} & & \| & & \| & & \downarrow{\scriptstyle x^{-1}[1]} \\
X & \xrightarrow{\bar{f}} & Y & \xrightarrow{g} & Z & \xrightarrow{\bar{h}} & X[1] \\
\| & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \| \\
X & \xrightarrow{\bar{f}'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{\bar{h}'} & X[1] \\
\downarrow{\scriptstyle x} & & \| & & \| & & \downarrow{\scriptstyle x[1]} \\
X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1]
\end{array}$$

It follows that $[1, y, z] \colon (f, g, h) \to (f', g', h')$ is the required diagram.

3.4. **C implies B.** Suppose that C holds and that we are given a diagram as in B. By TR1 there exists an exact triangle

$$Y \xrightarrow{\begin{pmatrix} y \\ -g \end{pmatrix}} Y' \oplus Z \xrightarrow{(a \ b)} C \xrightarrow{c} Y[1].$$

Fixing the triangle (f, g, h), we can apply C to get an exact commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow[g]{} & Z & \xrightarrow[h]{} & X[1] \\
\| & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle b} & & \| \\
X & \xrightarrow{f'} & Y' & \xrightarrow{a} & C & \xrightarrow{d} & X[1]
\end{array}$$

such that $c = f[1]d$. Note that we have used $yf = f'$. By TR3 and Lemma 1 there exists an isomorphism of triangles $[1, 1, \theta] \colon (f', a, d) \to (f', g', h')$. We set $z := \theta b$, so that $[1, y, z] \colon (f, g, h) \to (f', g', h')$ is a morphism of triangles. Moreover,

$$\theta a = g', \quad \theta b = z \quad \text{and} \quad c\theta^{-1} = f[1]d\theta^{-1} = f[1]h'.$$

Thus the following diagram is an isomorphism of triangles:

$$\begin{array}{ccccccc}
Y & \xrightarrow{\binom{y}{-g}} & Y' \oplus Z & \xrightarrow{(a\ b)} & C & \xrightarrow{c} & Y[1] \\
\| & & \| & & \downarrow{\scriptstyle\theta} & & \| \\
Y & \xrightarrow{\binom{y}{-g}} & Y' \oplus Y & \xrightarrow{(g'\ z)} & Z' & \xrightarrow{f[1]h'} & Y[1]
\end{array}$$

Since the top row is exact, so is the bottom row. Hence B holds.

Remark. The proofs are similar if we replace B by B'.

3.5. **Equivalence of B, D and D'.** Suppose that B holds and that we are given three exact triangles as in D'. We apply B to the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\| & & \downarrow{\scriptstyle u} & & & & \| \\
X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1]
\end{array}$$

to obtain a morphism of triangles $[1, u, u']$ such that the following triangle is exact:

$$Y \xrightarrow{\binom{u}{-g}} Y' \oplus Z \xrightarrow{(g'\ u')} Z' \xrightarrow{f[1]h'} Y[1].$$

We now apply B to the diagram

$$\begin{array}{ccccccc}
Y & \xrightarrow{\binom{u}{-g}} & Y' \oplus Z & \xrightarrow{(g'\ u')} & Z' & \xrightarrow{f[1]h'} & Y[1] \\
\| & & \downarrow{\scriptstyle (1\ 0)} & & & & \| \\
Y & \xrightarrow{u} & Y' & \xrightarrow{v} & W & \xrightarrow{w} & Y[1]
\end{array}$$

to obtain a morphism of triangles $[1, (1\ 0), v']$ such that the following triangle is exact:

$$Y' \oplus Z \xrightarrow{\begin{pmatrix} 1 & 0 \\ -g' & -u' \end{pmatrix}} Y' \oplus Z' \xrightarrow{(v\ v')} W \xrightarrow{\begin{pmatrix} u[1]w \\ -g[1]w \end{pmatrix}} Y'[1] \oplus Z[1].$$

By Lemma 5, we deduce that the following triangle is exact, where $w' := g[1]w$:

$$Z \xrightarrow{u'} Z' \xrightarrow{v'} W \xrightarrow{w'} Z[1].$$

Note that $wv' = f[1]h'$. Hence D' holds.

Clearly D' implies D, which in turn implies B.

3.6. **Equivalence of C and E.** Suppose that C holds and that we are given an exact triangle as in E:

$$Y \xrightarrow{\binom{u}{-g}} Y' \oplus Z \xrightarrow{(g'\ u')} Z' \xrightarrow{\delta} Y[1].$$

We may apply C to obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\| & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \| \\
X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X[1]
\end{array}$$

such that $\delta = f[1]h$. Moreover, we may fix beforehand either (f, g, h) or (f', g', h').

Similarly, we may rewrite the original triangle as

$$Y \xrightarrow{\binom{g}{u}} Z \oplus Y' \xrightarrow{(-u' \; g')} Z' \xrightarrow{\delta} Y[1].$$

We apply C again to obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} W[-1] & \xrightarrow{-w[-1]} & Y & \xrightarrow{-u} & Y' & \xrightarrow{-v} & W \\ \| & & \downarrow g & & \downarrow g' & & \| \\ W[-1] & \xrightarrow{-w'[-1]} & Y' & \xrightarrow{-u'} & Z' & \xrightarrow{-v'} & X[1] \end{array}$$

such that $\delta = wv'$. Moreover, we may fix beforehand either (u, v, w) or (u', v', w'). Hence E holds.

It is clear that E implies C.

References

[1] A. Neeman, *Triangulated Categories*, Annals of Mathematics Studies, 148 (Princeton University Press, Priceton, 2001).